ONE-DIMENSIONAL UNSTEADY SOLUTION OF THE EQUATION FOR THE KINETIC MOMENTS OF A MONATOMIC GAS

(ODNOMERNOE NESTATSIONARNOE RESHENIE URAVNENII KINETICHESKIKH MOMENTOV ODNOATOMNOGO GAZA)

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V.S. GALKIN
(Moscow)

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In [1] a class of exact solutions of the equations for the kinetic moments of a monatomic Maxwellian gas are studied, when external forces are absent, and the density ρ, the coefficient of viscosity μ, the pressure p, the stresses p_{ij} and all the remaining moments of the distribution function of higher order depend only on the time t^*, whilst the components of macroscopic velocity, moreover, depend linearly on the Cartesian coordinates x, y and z. The fundamental and most simple solutions of this class are the shear solution, by which the accuracy of the well known Chapman-Enskog method [2] and [5] of the relaxation kinetic equation [4] was studied, and also the one-dimensional time-dampted solution considered here. In this paper the basic attention to the applicability of the Chapman-Enskog method is given.

Let the velocity vector V of the gas be directed along the x-axis, and also

$$V = x / (t^* + c), \qquad c = \text{const} \tag{1}$$

Making use of the relation obtained in [1] for the stated class of flows, we find that

$$\rho = \rho(0) / (1 + t), \qquad t = t^* / c \tag{2}$$

$$\frac{p_{xy}}{p_{xy}(0)} = \frac{p_{xz}}{p_{xz}(0)} = (1 + t)^{-(2+1/\beta)}, \qquad \frac{p_{yz}}{p_{yz}(0)} = (1 + t)^{-(1+1/\beta)}$$

$$\beta = \frac{\mu(0)}{cp(0)} = \frac{5}{3}\frac{M^2}{R} \tag{3}$$

$$p_{yy} = -{}^1/_2 p_{xx} + [p_{yy}(0) + {}^1/_2 p_{xx}(0)](1 + t)^{-(1+1/\beta)}, \qquad p_{xx} + p_{yy} + p_{zz} = 0$$

Here M and R are characteristic values of the Mach and Reynolds numbers. The equation of energy and the equation for p_{xx} have, respectively, the form

$$\frac{dp}{d\eta} + 5p + 2p_{xx} = 0, \qquad \frac{dp_{xx}}{d\eta} + 4p + \left(7 + \frac{3}{\beta}\right)p_{xx} = 0, \qquad \eta = \frac{\ln(t^* + C)}{3} \tag{4}$$

Let us introduce the notation

$$\Pi = p / p(0), \qquad \Pi_{xx} = p_{xx} / p \qquad \text{or} \qquad \Pi_{xx}(0) = p_{xx}(0) / p(0) \tag{5}$$

The solution of the system (4) will have the form

$$\Pi = A\,(1+t)^{r_1/3}\,[1+B\,(1+t)^k], \quad A = \frac{2}{r_2-r_1}\left[\frac{5+r_2}{2} + \Pi_{xx}(0)\right], \quad B = \frac{1}{A} - 1$$

$$\Pi_{xx} = -\frac{5+r_1}{2}\,\Pi_{xx}^{\bullet}, \qquad \Pi_{xx}^{\bullet} = [1+B\,(1+t)^k]^{-1}\left[1 + B\,\frac{5+r_2}{5+r_1}\,(1+t)^k\right] \qquad (6)$$

$$r_{1,2} = -\frac{3}{2\beta}\,[1+4\beta \mp \sqrt{1+4\beta\,(1/3+\beta)}], \qquad k = \frac{r_2-r_1}{3}$$

For small β and fixed values of $\Pi_{xx}(0)$ we shall have

$$r_1 = -5 + \frac{8}{3}\beta - \frac{16}{9}\beta^2 - \frac{32}{27}\beta^3 + \frac{320}{81}\beta^4 + O(\beta^5)$$

$$r_2 - r_1 = -3\beta^{-1} - 2 + O(\beta), \; A = 1 - \frac{2}{3}\beta\Pi_{xx}(0) + O(\beta^2) \qquad (7)$$

For any fixed values of t, $\Pi_{xx}(0)$ and $\beta \to 0$, from the relation (6) with allowance for (7) we obtain the zero order terms in the asymptotic expansion of the exact solution for small β

$$\Pi = (1+t)^{-5/3}, \qquad \Pi_{xx} = -{}^4/_3\beta \qquad (8)$$

which coincide with the solution of the problem in the approximations of Euler and Navier-Stokes and do not depend upon $\Pi_{xx}(0)$

The solution of the problem by the method of Chapman-Enskog has the form

$$\Pi = (1+t)^{1/r}$$
$$\Pi_{xx} = -{}^1/_2\,(5+r) \qquad (9)$$

where r coincides with the expansion r_1 (Formula (7)); in the Navier-Stokes approximation

$$r \equiv r^{(1)} = -5 + {}^8/_3\beta$$

in Barnett approximation

$$r \equiv r^{(2)} = -5 + {}^8/_3\beta - {}^{16}/_9\beta^2$$

in the third approximation, calculated with the help of the iterational method of [5],

$$r \equiv r^{(3)} = r^{(2)} - {}^{32}/_{27}\beta^3$$

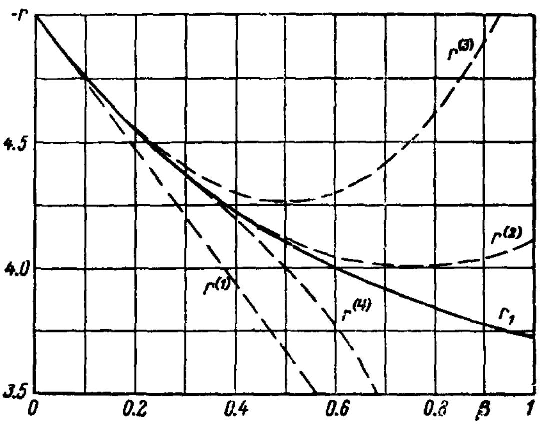

Fig. 1

In Fig. 1 the exact values of r_1 are compared with the approximations

$$(r^{(4)} = r^{(3)} + {}^{320}/_{81}\beta^4)$$

Barnett's approximation is significantly more accurate than the Navier-Stokes approximation, and agrees well with the exact solution even in the region of divergence of the Chapman-Enskog method. By the same method as in Section 36 of [2], where the shear flow is cosidered, it is easy to show that the series for Π_{xx} obtained by this method converges at least when $\beta < (\sqrt{10} - 1)/6$ and consequently the corresponding series for r in terms of β converges to r_1 .

The approximate solution (9) differs from the exact solution not only that r is an approximate value for r_1 , but differs also in its structure. If $A = 1$, then $B = 0$ and $\Pi_{xx}(0) = -(5 + r_1)/2$. Only for such a value of $\Pi_{xx}(0)$ the solution by the Chapman-Enskog method have the same structure as the exact solution and the small values of β converges to it. Of course, each approximation of this method is applicable only in a finite interval of the value of t , since when $t \to \infty$ the approximate values of Π can differ appreciably from the exact values.

For arbitrary values of $\Pi_{xx}(0)$ the solution by the Chapman-Enskog method has an asymptotic character, giving correctly only the zero order terms of the solution with respect to β (8), since even in the first term of the expansion of A with respect to β the ratio $\Pi_{xx}(0)$ occurs. It is necessary to remark that, in contrast to the Chapman-Enskog method, in the kinetic theory we assume a certain arbitrariness in the initial values of $\tau_{ij} = p_{ij} + p\delta_{ij}$ and the other moments of higher order, but the degree of this arbitrariness is unknown. An essential limitation consists in the positiveness of the distribution function and, consequently, of its even moments.

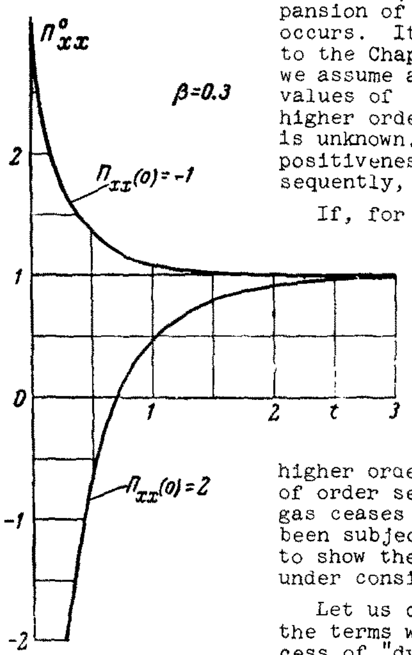

Fig. 2

If, for example, $p_{yy}(0) = p_{zz}(0) = -\frac{1}{2}p_{xx}(0)$, then, taking account of the conditions $\tau_{yy} > 0$ and $\tau_{xx} > 0$, we have $-1 < \Pi_{xx}(0) < 2$.

For the final answer to this question it is necessary to find the distribution function or the complete spectrum of its moments.

In the Chapman-Enskog method it is assumed that the fluctuation of the initial values of the moments of the distribution function of second and higher order, die out, and after a period of time of order several relaxation times the state of the gas ceases to depend upon them. This assumption has been subject to criticism; it is particularly easy to show the falseness of this assumption in the case under consideration here.

Let us consider the case of large t. Discarding the terms which die out rapidly with time (this process of "dying out" is illustrated in Fig. 2), from Formulas (3) we obtain

$$p_{ij}=0 \ (i \neq j), \quad p_{yy} = p_{zz} = -\tfrac{1}{2}p_{xx},$$

which coincides with the corresponding results of the Chapman-Enskog method. From Formulas (6) we obtain

$$\Pi / A = (1 + t)^{1/2 r_1}, \qquad \Pi_{xx} = -\tfrac{1}{2}(5 + r_1)$$

Accordingly, for large t, *small* β *and arbitrary* $\Pi_{xx}(0)$ *the Chapman-*Enskog method gives correctly the values of the ratios Π/A and Π_{xx} and incorrectly the absolute values of p and p_{xx} (with the exclusion of the case $A = 1$, considered above).

The conclusions drawn above are valid also in the case of shear flow with the exception of the circumstance that when $A = 1$ in the solution considered here, the results of the calculations in the Barnett approximation are close to exact ones in a considerably wider range of values of β than in the case of shear flow. The relative success of the Chapman-Enskog method in the problem of shear flow in [2] is explained by the fact that in this case those terms of the equations of this method are identically equal to zero, the presence of which, as is established in [2] and [5], throw doubt on the correctness of the Chapman-Enskog method. However, in the case considered here these terms are not zero.

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